

NEGATIVE ENERGY BLOWUP RESULTS FOR THE FOCUSING HARTREE HIERARCHY VIA IDENTITIES OF VIRIAL AND LOCALIZED VIRIAL TYPE

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ABSTRACT. We establish virial and localized virial identities for solutions to the Hartree hierarchy, an infinite system of partial differential equations which arises in mathematical modeling of many body quantum systems. As an application, we use arguments originally developed in the study of the nonlinear Schrödinger equation (see work of Zakharov, Glassey, and Ogawa–Tsutsumi) to show that certain classes of negative energy solutions must blow up in finite time.

The most delicate case of this analysis is the proof of negative energy blowup without the assumption of finite variance; in this case, we make use of the localized virial estimates, combined with the quantum de Finetti theorem of Hudson and Moody and several algebraic identities adapted to our particular setting. Application of a carefully chosen truncation lemma then allows for the additional terms produced in the localization argument to be controlled.

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1. INTRODUCTION

Fix $d \geq 1$, let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be given, and let $\Delta_{\pm}^{(k)}$ be the operator given by

$$\Delta_{\pm}^{(k)} := \sum_{i=1}^k \Delta_{x_i} - \Delta_{x'_i}.$$

In the present work, we consider an infinite system of coupled PDEs, often referred to as the *Hartree hierarchy*,

$$i\partial_t \gamma^{(k)} + \Delta_{\pm}^{(k)} \gamma^{(k)} = \mu \sum_{j=1}^k B_{j,k+1}^{\pm} \gamma^{(k+1)}, \quad (1.1)$$

for functions $(t, \mathbf{x}, \mathbf{x}') \mapsto \gamma^{(k)}(t, \mathbf{x}, \mathbf{x}')$, $k \geq 1$, where the variables t , \mathbf{x} , and \mathbf{x}' belong, respectively, to a time interval $I \subset \mathbb{R}$ and the spaces \mathbb{R}^{dk} and \mathbb{R}^{dk} . Here, we take $\mu \in \{-1, 1\}$, and

$$B_{j,k+1}^{\pm} \gamma^{(k+1)} = B_{j,k+1}^{+} \gamma^{(k+1)} - B_{j,k+1}^{-} \gamma^{(k+1)},$$

with $B_{j,k+1}^{\pm}$ defined by

$$\begin{aligned} (B_{j,k+1}^{+} \gamma^{(k+1)})(t, \mathbf{x}, \mathbf{x}') &:= \int \gamma^{(k+1)}(\mathbf{x}, y, \mathbf{x}', y) V(x_j - y) dy, \\ (B_{j,k+1}^{-} \gamma^{(k+1)})(t, \mathbf{x}, \mathbf{x}') &:= \int \gamma^{(k+1)}(\mathbf{x}, y, \mathbf{x}', y) V(x'_j - y) dy, \end{aligned}$$

under the notational conventions $\mathbf{x} = (x_1, \dots, x_k)$ and $\mathbf{x}' = (x'_1, \dots, x'_k)$, where x_i and x'_i are vectors in \mathbb{R}^d for $1 \leq i \leq k$.

In what follows, we consider solutions $(\gamma^{(k)})_{k \geq 1}$ to (1.1) satisfying the initial condition

$$\gamma^{(k)}(0, \mathbf{x}, \mathbf{x}') = \gamma_0^{(k)}(\mathbf{x}, \mathbf{x}').$$

The equation (1.1) enjoys a special relationship with the classical Hartree equation (that is, the nonlinear Schrödinger equation with nonlocal convolution-type nonlinearity). In particular, when the initial data $(\gamma_0^{(k)})$ takes the form

$$\gamma_0^{(k)}(\mathbf{x}, \mathbf{x}') = \prod_{j=1}^k \phi_0(x_j) \overline{\phi_0(x'_j)}, \quad k \geq 1,$$

with $\phi_0 \in H^2(\mathbb{R}^d)$, the function

$$\gamma^{(k)}(t, \mathbf{x}, \mathbf{x}') := \prod_{j=1}^k \phi(t, x_j) \overline{\phi(t, x'_j)} \quad (1.2)$$

is a particular solution of (1.1) provided that $t \mapsto \phi(t)$ is a solution of the Hartree equation

$$i\partial_t \phi + \Delta \phi = \mu(V * |\phi|^2)\phi \quad (1.3)$$

with $\phi(0) = \phi_0$. Initial conditions of the form (1.2) are sometimes termed *factorized* initial data. In accordance with the usual nomenclature for (1.3) (and, more generally, nonlinear Schrödinger equations), we will say that the hierarchy (1.1) has *defocusing* nonlinearity when $\mu = 1$, and *focusing* nonlinearity when $\mu = -1$.

This infinite system of equations arises in the study of the mean field limit of quantum mechanical systems, where the pairwise interactions between quantum particles are governed by the potential V (see for instance [16]). When V is taken to be the Dirac measure $\delta(x)$, (1.1) becomes the *Gross-Pitaevskii (GP) hierarchy* (see, e.g. [7, 8, 14] as well as [4, 5, 2, 3, 13, 16] and the references cited therein).

In the present work, we establish a class of identities for solutions to (1.1) which hold in analogy to the usual virial identities for nonlinear Schrödinger equations. As an application of these ideas, we invoke the classical argument of Glassey [9] to show that certain classes of negative energy solutions must blow up in finite time, under the assumption of initially finite variance (see Theorem 1.1 below). Moreover,

under certain hypotheses on the potential V , we establish negative energy blowup in the absence of the finite variance assumption. This result is in the spirit of a result due to Ogawa and Tsutsumi [15] for the nonlinear Schrödinger equation; see also work of Hirata [10] for a related result for the Hartree equation. The main tool (and the main novelty of the present work) is the derivation of a class of localized virial identities, the derivation of which in the hierarchy setting produces a number of additional terms which must be dealt with carefully.

We now prepare some notation to formally state our main results. Throughout the paper, we will make frequent use of the trace operator, given for $f : \mathbb{R}^{dk} \times \mathbb{R}^{dk} \rightarrow \mathbb{C}$ by

$$\mathrm{Tr}(f) := \int f(\mathbf{x}, \mathbf{x}) d\mathbf{x}.$$

with $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^{dk}$ as before, as well as the partial trace in the last variable,

$$\mathrm{Tr}_k(f)(x_1, \dots, x_{k-1}, x'_1, \dots, x'_{k-1}) := \int f(\mathbf{x}, \mathbf{x}) dx_k$$

for $k \geq 1$, and use the notational convention

$$\gamma^{(k)}(t, \mathbf{x}, \mathbf{x}') = \gamma^{(k)}(\mathbf{x}, \mathbf{x}'), \quad t \in I,$$

omitting explicit specification of the time variable when there is no potential for confusion.

We say that a solution $(\gamma^{(k)})_{k \geq 1}$ to (1.1) (in the sense of the integral Duhamel formulation) is “(A)–(D)-admissible” if it satisfies the following properties for all $k \in \mathbb{N}$:

(A) $\gamma^{(k)} \in C(I; H^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk}))$, $\mathrm{Tr} \gamma^{(k)} = 1$, and $\gamma^{(k)} \succeq 0$, in the sense that

$$\left\langle \psi, \int_{\mathbb{R}^d} \gamma^{(k)}(\mathbf{x}, \cdot) \psi(\mathbf{x}) d\mathbf{x} \right\rangle_{L^2_{\mathbf{x}'}(\mathbb{R}^{dk})} \geq 0 \quad \text{for every } \psi \in L^2,$$

(B) $\gamma^{(k)}$ is symmetric with respect to permutations of the variables \mathbf{x} and permutations of the variables \mathbf{x}' : for every $\sigma \in S_k$ let P_σ denote the map $(x_1, x_2, \dots, x_k) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})$. Then for every $\sigma, \tau \in S_k$, one has

$$\gamma^{(k)}(t, \mathbf{x}, \mathbf{x}') = \gamma^{(k)}(t, P_\sigma(\mathbf{x}), P_\tau(\mathbf{x}')).$$

(C) $\gamma^{(k)}$ is Hermitian:

$$\gamma^{(k)}(t, \mathbf{x}, \mathbf{x}') = \overline{\gamma^{(k)}(t, \mathbf{x}', \mathbf{x})},$$

and,

(D) $\gamma^{(k)}$ is admissible:

$$\gamma^{(k)}(t) = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}(t)), \text{ that is,}$$

$$\gamma^{(k)}(t, \mathbf{x}, \mathbf{x}') = \int \gamma^{(k+1)}(t, \mathbf{x}, x_{k+1}, \mathbf{x}', x_{k+1}) dx_{k+1}.$$

The properties (A)–(D) are preserved under the evolution—that is, if the sequence of initial data $(\gamma_0^{(k)})_{k \geq 1}$ satisfies (A)–(D) as functions of \mathbf{x} and \mathbf{x}' , then the corresponding solution is (A)–(D)-admissible. For treatment of this invariance in the case of the Gross–Pitaevskii hierarchy, see Section 5 and Appendix B in [6]; we remark that the arguments for (1.1) are similar.

As we will see in Section 2.1 below, (A)–(D)-admissible solutions to (1.1) obey conservation of two relevant quantities: the mass $\text{Tr}(\gamma_0^{(1)}(t))$, and the energy

$$E_1 = E_1(t) := -\frac{1}{2} \text{Tr}(\Delta_{x_1} \gamma^{(1)}(t)) + \frac{\mu}{4} \text{Tr}(B_{1,2}^+ \gamma^{(2)}(t)). \quad (1.4)$$

These conserved quantities play a fundamental role in the analysis of long-time and global properties of the evolution (see also [4, 5] for related results concerning the GP hierarchy). Indeed, when the sign μ of the nonlinearity is positive, the conserved energy E_1 gives uniform-in-time control over each of its component terms; in a variety of settings, this information is sufficient to conclude that solutions exist globally in time. On the other hand, when μ is negative, no such control is guaranteed. Indeed, as shown in the work of Zakharov [17] and Glassey [9], in this (focusing) case negative initial energy leads to finite-time blowup results for the nonlinear Schrödinger equation, under an assumption of initially finite variance (see also [1] for a comprehensive treatment of these and related results).

Our first theorem is a variant of Glassey’s argument, adapted to the hierarchy (1.1). To state this result, we will make use of the quantity

$$V_1(t) = \text{Tr}(|x|^2 \gamma^{(1)}(t)).$$

for solutions $(\gamma^{(k)})_{k \geq 1}$ defined on a time interval $I \subset \mathbb{R}$, and $t \in I$.

Theorem 1.1. *Fix $\mu = -1$ and suppose that $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is an even function such that*

$$V + \frac{1}{2} x \cdot \nabla V \leq 0.$$

Let $(\gamma^{(k)})_{k \geq 1}$ be an (A)–(D)-admissible solution to (1.1) defined on an interval $I \subset \mathbb{R}$ with

$$V_1(0) < \infty \quad \text{and} \quad E_1 < 0.$$

Then I is bounded.

In view of Theorem 1.1, it is natural to consider whether the finite variance condition $V_1(0) < \infty$ can be relaxed. For the nonlinear Schrödinger equation, a partial result in this direction has been given by Ogawa and Tsutsumi [15] (see also work of Hirata [10] for a related result concerning certain instances of the Hartree equation). In our setting, we prove the following theorem:

Theorem 1.2. *Fix $\mu = -1$ and suppose that $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is an even function with $V \geq 0$,*

$$V + \frac{1}{2} x \cdot \nabla V \leq 0, \quad (1.5)$$

$$\sup_{|x| \geq R} |x| |(\nabla V)(x)| \rightarrow 0 \quad (1.6)$$

as $R \rightarrow \infty$, and

$$\frac{\| |x| |\nabla V(x)| \|_{L^1(|x| \geq R^{1/2})}}{R^{(N-1)/2}} \rightarrow 0 \quad (1.7)$$

as $R \rightarrow \infty$.

Then, if $(\gamma^{(k)})_{k \geq 1}$ is any (A)–(D)-admissible solution to (1.1) defined on an interval $I \subset \mathbb{R}$ with $\gamma_0^{(1)}(x, x')$ radial in x and x' (in the sense that $\gamma_0^{(1)}(x, x') = \gamma_0^{(1)}(|x|, |x'|)$) and $\gamma_0^{(2)}(x, y, x', y')$ radial in each of x, x', y and y' , then the condition

$$E_1 < 0$$

implies I is bounded, where E_1 is as defined in (1.4).

As we briefly described above, the proof of Theorem 1.2 is based on a localized form of the virial identities used to prove Theorem 1.1. We establish these identities in Proposition 3.3, with some additional algebraic tools useful in our arguments in the subsequent Lemma 3.4 and Lemma 4.1.

The analysis leading to the proof of this result consists in the application of essentially three main ingredients:

- (1) the formulation of an appropriate form of a radial truncation lemma, described in Section 2.2; this truncation lemma allows for the additional terms appearing in the localized identities to be controlled by the conserved energy,
- (2) the quantum de Finetti theorem of [11] (see also [12], [3], and the references cited therein); this provides the algebraic structure necessary to estimate the “decoupled” nonlinearity via standard convolution estimates, and
- (3) decay properties of the functions $\gamma^{(k)}$ arising from the Strauss lemma, ensured by our radiality assumption.

We conclude this introduction with a brief outline of the rest of the paper. In section 2, we establish some further notational conventions, establish conservation of mass and energy for (1.1) (note that similar conservation laws were obtained by Chen, Pavlovic and Tzirakis in [4] for the related Gross-Pitaevskii hierarchy), and formulate and prove the truncation lemma which will be used in our proof of Theorem 1.2, the result on negative energy blowup without an assumption of finite variance. Section 3 is then devoted to the derivation of the relevant virial and localized virial identities which form the basis of our subsequent analysis. The proofs of Theorem 1.1 and Theorem 1.2 are then given in Section 4.

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2. NOTATION AND PRELIMINARIES

We begin by specifying our conventions for the Fourier transform. For $k \geq 1$ and $\gamma^{(k)} \in L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$, we take the Fourier transform in x_i and x'_i , $i = 1, \dots, k$

to be defined by, respectively

$$\begin{aligned}\mathcal{F}_{x_i}[\gamma^{(k)}](x_1, \dots, x_{i-1}, \xi_i, x_{i+1}, \dots, x_k, x'_1, \dots, x'_k) \\ = \int e^{-ix_i \xi_i} \gamma^{(k)}(x_1, \dots, x_k, x'_1, \dots, x'_k) dx_i, \\ \mathcal{F}_{x'_i}[\gamma^{(k)}](x_1, \dots, x_k, x'_1, \dots, x'_{i-1}, \xi'_i, x'_{i+1}, \dots, x'_k) \\ = \int e^{ix'_i \xi'_i} \gamma^{(k)}(x_1, \dots, x_k, x'_1, \dots, x'_k) dx'_i.\end{aligned}$$

With this notation, we have the standard identities $\nabla_{x_i} \gamma^{(k)} = [(i\xi_i) \widehat{\gamma^{(k)}}]^\vee$ and $\nabla_{x'_i} \gamma^{(k)} = [(-i\xi'_i) \widehat{\gamma^{(k)}}]^\vee$ for $1 \leq i \leq k$.

In the remainder of this section, we collect some preliminary results concerning (1.1). In particular, we show that the hierarchy enjoys conservation of two quantities which are in analogy to mass and energy functionals for the classical Hartree equation. We also establish a fundamental truncation lemma which will play an important role in the proof of Theorem 1.2.

2.1. Conservation of mass and energy. We now establish conservation of the mass, $\text{Tr}(\gamma^{(1)}(t))$, and the energy, $E_1(t)$ (given by (1.4)), associated to (1.1). As a preliminary comment, we recall an identity from [4] (see, e.g. (4.10)–(4.11) in [4]): fix $d \geq 1$ and let $A : (x_1, x'_1) \mapsto A(x_1, x'_1) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ be given; then

$$\text{Tr}(\Delta_{x_1} A) = \text{Tr}(\Delta_{x'_1} A) = \text{Tr}(-\nabla_{x_1} \cdot \nabla_{x'_1} A). \quad (2.1)$$

We begin with the conservation of mass, corresponding to invariance of $\text{Tr}(\gamma^{(1)}(t))$ under the evolution.

Proposition 2.1 (Conservation of mass for (1.1)). *Suppose that $(\gamma^{(k)})_{k \geq 1}$ is an (A)–(D)-admissible solution to the Hartree hierarchy (1.1). Then we have*

$$\partial_t \text{Tr}(\gamma^{(1)}(t)) = 0.$$

Proof. By (1.1), we get

$$\begin{aligned}\partial_t \text{Tr}(\gamma^{(1)}) &= \int \partial_t \gamma^{(1)}(x, x) dx \\ &= i \int (\Delta_{x_1} \gamma^{(1)})(x, x) - (\Delta_{x'_1} \gamma^{(1)})(x, x) dx - \mu i \int B_{1,2}^\pm \gamma^{(2)}(x, x) dx.\end{aligned}$$

Now, invoking (2.1) to see that the first integral vanishes, and writing out the definition of the operator $B_{1,2}^\pm$, the right side of the above equality is equal to

$$-\mu i \int \gamma^{(2)}(x, y, x, y) \left[V(x - y) - V(x' - y) \right] \Big|_{(x, x') = (x, x)} dy dx = 0,$$

as desired. \square

We now establish a conservation law for the energy functional E_1 defined in (1.4).

Proposition 2.2 (Conservation of energy for (1.1)). *Suppose that $(\gamma^{(k)})_{k \geq 1}$ is an (A)–(D)-admissible solution to the Hartree hierarchy (1.1). Then we have*

$$\partial_t[E_1(t)] = \partial_t \left[-\frac{1}{2} \text{Tr}(\Delta_{x_1} \gamma^{(1)}(t)) + \frac{\mu}{4} \text{Tr}(B_{1,2}^+ \gamma^{(2)}(t)) \right] = 0.$$

Proof. Since $(\gamma^{(k)})$ solves (1.1), we have

$$\begin{aligned} & \partial_t \left[\frac{1}{2} \text{Tr}(-\Delta_{x_1} \gamma^{(1)}(t)) + \frac{\mu}{4} \text{Tr}(B_{1,2}^+ \gamma^{(2)}(t)) \right] \\ &= \frac{i}{2} \text{Tr} \left(-\Delta_{x_1} \left[\Delta_{x_1} \gamma^{(1)} - \Delta_{x'_1} \gamma^{(1)} - \mu B_{1,2}^\pm \gamma^{(2)} \right] \right) \\ & \quad + \frac{\mu i}{4} \text{Tr} \left(B_{1,2}^+ \left[\Delta_{\pm}^{(2)} \gamma^{(2)} - \mu \sum_{j=1}^2 B_{j,3}^\pm \gamma^{(3)} \right] \right). \end{aligned} \quad (2.2)$$

We now observe that, taking the Fourier transform and its inverse, and applying Fubini's theorem, one has the identity

$$\begin{aligned} & \text{Tr} \left(-\Delta_{x_1} \left[\Delta_{x_1} \gamma^{(1)}(t) - \Delta_{x'_1} \gamma^{(1)}(t) \right] \right) \\ &= \int e^{ix_1 \cdot (\xi_1 - \xi'_1)} |\xi_1|^2 (|\xi'_1|^2 - |\xi_1|^2) \widehat{\gamma^{(1)}}(\xi_1, \xi'_1) d\xi_1 d\xi'_1 dx_1 \\ &= \int \delta(\xi_1 - \xi'_1) |\xi_1|^2 (|\xi'_1|^2 - |\xi_1|^2) \widehat{\gamma^{(1)}}(\xi_1, \xi'_1) d\xi_1 d\xi'_1 = 0, \end{aligned}$$

while expansion of the definitions of the operators $B_{1,2}^+$ and $B_{j,3}^\pm$ gives

$$\begin{aligned} & \text{Tr} \left(B_{1,2}^+ \left[\sum_{j=1}^2 B_{j,3}^\pm \gamma^{(3)} \right] \right) \\ &= \int \int \left[\sum_{j=1}^2 \int \gamma^{(3)}(x_1, x_2, z) (V(x_j - z) - V(x_j - z)) dz \right] V(x_1 - x_2) dx_2 dx_1 \\ &= 0. \end{aligned}$$

Substituting these identities into the right side of (2.2), we obtain that $\partial_t[E(t)]$ is equal to

$$\begin{aligned} & -\frac{\mu i}{2} \text{Tr} \left(\Delta_{x_1} \left[\int \gamma^{(2)}(x_1, y, x'_1, y) (V(x_1 - y) - V(x'_1 - y)) dy \right] \right) \\ & + \frac{\mu i}{4} \text{Tr} \left(\int [(\Delta_{x_1} \gamma^{(2)})(x_1, y, x_1, y) - (\Delta_{x'_1} \gamma^{(2)})(x_1, y, x'_1, y)] V(x_1 - y) dy \right) \\ & + \frac{\mu i}{4} \text{Tr} \left(\int [(\Delta_{x_2} \gamma^{(2)})(x_1, y, x_1, y) - (\Delta_{x'_2} \gamma^{(2)})(x_1, y, x'_1, y)] V(x_1 - y) dy \right). \end{aligned} \quad (2.3)$$

Next, using (2.1) and the symmetry of $\gamma^{(2)}$ with respect to permutations of the variables, and observing that one has the equalities

$$\Delta_{x'_1} [\gamma^{(2)}(x_1, y, x'_1, y) V(x_1 - y)] = (\Delta_{x'_1} \gamma^{(2)})(x_1, y, x'_1, y) V(x_1 - y),$$

and

$$\Delta_{x_1} [\gamma^{(2)}(x_1, y, x'_1, y) V(x'_1 - y)] = (\Delta_{x_1} \gamma^{(2)})(x_1, y, x'_1, y) V(x'_1 - y),$$

we get

$$\begin{aligned}
(2.3) = & -\frac{\mu i}{2} \operatorname{Tr} \left(\int \Delta_{x'_1} [\gamma^{(2)}(x_1, y, x'_1, y) V(x_1 - y)] dy \right) \\
& + \frac{\mu i}{2} \operatorname{Tr} \left(\int \Delta_{x_1} [\gamma^{(2)}(x_1, y, x'_1, y) V(x'_1 - y)] dy \right) \\
& + \frac{\mu i}{2} \operatorname{Tr} \left(\int (\Delta_{x_1} \gamma^{(2)})(x_1, y, x'_1, y) V(x_1 - y) dy \right) \\
& - \frac{\mu i}{2} \operatorname{Tr} \left(\int (\Delta_{x'_1} \gamma^{(2)})(x_1, y, x'_1, y) V(x_1 - y) dy \right) = 0.
\end{aligned}$$

This completes the proof of the proposition. \square

Remark 2.3. By the symmetry of $\gamma^{(k)}$, $k \geq 1$ with respect to permutations of the variables, Proposition 2.2 immediately implies that the quantities

$$\begin{aligned}
E_k(t) &:= \frac{1}{2} \operatorname{Tr} \left(\sum_{j=1}^k -\Delta_{x_j} \gamma^{(k)}(t) \right) + \frac{\mu}{4} \operatorname{Tr} \left(\sum_{j=1}^k B_{j,k+1}^+ \gamma^{(k+1)}(t) \right) \\
&= k \left(\frac{1}{2} \operatorname{Tr} (\gamma^{(1)}(t)) + \frac{\mu}{4} \operatorname{Tr} (B_{1,2}^+ \gamma^{(2)}(t)) \right), \quad k \geq 2,
\end{aligned}$$

are also conserved. A similar family of conserved quantities was observed in [4] for the Gross-Pitaevskii hierarchy.

2.2. Truncation lemma. In this subsection, we recall some elementary cutoff function bounds which will be useful in our proof of Theorem 1.2. For some related estimates used in the analysis of the Hartree equation, see [10] and the references cited there.

Fix $\epsilon > 0$, and choose $\rho \in C_c^2(\mathbb{R})$ such that $\rho(x) \geq 0$ for all $x \in \mathbb{R}$, $\operatorname{supp}(\rho) \subset (1, 3)$ with $\rho > 0$ on $(\frac{5}{4}, \frac{11}{4})$, $\int \rho(x) dx = 1$, $\rho' \geq 0$ on $(1, \frac{3}{2})$, and $\rho(x) = \rho(4 - x)$ for all $x \in \mathbb{R}$. Define $\psi : [0, \infty) \rightarrow \mathbb{R}$ by

$$\psi(x) = x - \int_0^x (x - y) \rho(y) dy$$

for $x \geq 0$, and, for all $R \geq 0$, $F_R : [0, \infty) \rightarrow \mathbb{R}$ by

$$F_R(r) := \int_0^{r^2/R} \rho(s) ds \quad (2.4)$$

for $r \geq 0$.

Lemma 2.4. *For each $R \geq 1$ there exists $C > 0$ such that for every $x, y \in \mathbb{R}^d$ with $\max\{|x|, |y|\} \geq R^{1/2}$ and $|x - y| \leq R^{1/2}$, we have*

$$\begin{aligned}
& \left| (x - y) - \left(\psi' \left(\frac{|x|^2}{R} \right) x - \psi' \left(\frac{|y|^2}{R} \right) y \right) \right| \\
& \lesssim \left(F_R(|x|) + \frac{|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) + F_R(|y|) + \frac{|y|^2}{R} \rho \left(\frac{|y|^2}{R} \right) \right) |x - y|.
\end{aligned} \quad (2.5)$$

Proof. Since $\psi'(t) = 1 - \int_0^t \rho(s) ds$ for $t \geq 0$, we may define

$$f(t) := (x + t(y - x)) F_R(|x + t(y - x)|)$$

for $0 \leq t \leq 1$, and observe that with this notation the left side of (2.5) becomes $|f(1) - f(0)|$. To estimate this quantity, we write

$$|f(1) - f(0)| = \left| \int_0^1 f'(s) ds \right| \leq \int_0^1 |f'(s)| ds \quad (2.6)$$

and estimate

$$\begin{aligned} |f'(s)| &\leq |y - x| \sup_s |F_R(|x + s(y - x)|)| \\ &\quad + \sup_s |(x + s(y - x))| \frac{d}{ds} [F_R(|x + s(y - x)|)]. \end{aligned} \quad (2.7)$$

Now, since $|x + s(y - x)| \leq \max\{|x|, |y|\}$ holds for $0 \leq s \leq 1$, it follows from the observation that F_R is increasing on $[0, \infty)$ that

$$F_R(|x + s(y - x)|) \leq \max\{F_R(|x|), F_R(|y|)\} \leq F_R(|x|) + F_R(|y|). \quad (2.8)$$

On the other hand, explicit computation gives

$$\frac{d}{ds} [F_R(|x + s(y - x)|)] = \rho(|x + s(y - x)|^2/R) \frac{2(x + s(y - x)) \cdot (y - x)}{R}, \quad (2.9)$$

so that (combining (2.6), (2.7), and (2.8) with (2.9)) we obtain

$$\begin{aligned} |f(1) - f(0)| &\leq \left(F_R(|x|) + F_R(|y|) \right. \\ &\quad \left. + \sup_s \frac{2|x + s(y - x)|^2}{R} \rho(|x + s(y - x)|^2/R) \right) |x - y|. \end{aligned} \quad (2.10)$$

To estimate the supremum, We now consider three cases depending on the sizes of $|x|^2/R$ and $|y|^2/R$.

Case 1: Suppose first that $|x|^2/R < \frac{3}{2}$ and $|y|^2/R < \frac{3}{2}$. Invoking once again $|x + s(y - x)| \leq \max\{|x|, |y|\}$ for $s \in [0, 1]$, we then have

$$\frac{|x + s(y - x)|^2}{R} \leq \max \left\{ \frac{|x|^2}{R}, \frac{|y|^2}{R} \right\} \leq \frac{3}{2}$$

for all such s . Since ρ is increasing on $(1, \frac{3}{2})$, this gives

$$\begin{aligned} \sup_s \frac{|x + s(y - x)|^2}{R} \rho \left(\frac{|x + s(y - x)|^2}{R} \right) &\leq \max \left\{ \frac{|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right), \frac{|y|^2}{R} \rho \left(\frac{|y|^2}{R} \right) \right\} \\ &\leq \frac{|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) + \frac{|y|^2}{R} \rho \left(\frac{|y|^2}{R} \right) \end{aligned}$$

which leads to the desired bound in this case.

Case 2: We now consider the case when either (i) $|x|^2/R < \frac{3}{2}$ and $|y|^2/R \geq \frac{3}{2}$; or (ii) $|x|^2/R \geq \frac{3}{2}$ and $|y|^2/R < \frac{3}{2}$. Since the estimate obtained is symmetric in x and y , we may assume that we are in case (i) without any loss of generality. In this setting, $|x + s(y - x)| \leq \max\{|x|, |y|\} = |y|$ for $s \in [0, 1]$ gives

$$\sup_s \frac{|x + s(y - x)|^2}{R} \rho \left(\frac{|x + s(y - x)|^2}{R} \right) \leq \frac{|y|^2}{R} \|\rho\|_{L^\infty}.$$

Noting that the triangle inequality and the hypothesis $|x - y| \leq R^{1/2}$ imply

$$\frac{|y|^2}{R} \leq \frac{(|y - x| + |x|)^2}{R} \leq (1 + \sqrt{3/2})^2,$$

and observing that $|y|^2/R \geq \frac{3}{2}$ implies

$$F_R(|y|) \geq \int_0^{3/2} \rho ds =: c_0(\rho) > 0, \quad (2.11)$$

we get the bound

$$\sup_s \frac{|x + s(y - x)|^2}{R} \rho \left(\frac{|x + s(y - x)|^2}{R} \right) \leq C \|\rho\|_{L^\infty} \leq \frac{C \|\rho\|_{L^\infty}}{c_0(\rho)} F_R(|y|)$$

which again leads to the desired inequality.

Case 3: Suppose now that $|x|^2/R \geq \frac{3}{2}$ and $|y|^2/R \geq \frac{3}{2}$. In this case we note that, in view of the assumption $|x - y| \leq R^{1/2}$, the condition $|x|^2/R \geq 10$ implies $\rho(|x + s(y - x)|^2/R) = 0$ for all $0 \leq s \leq 1$. Indeed, if $|x|^2/R \geq 10$, we have

$$|x + s(y - x)| \geq |x| - |y - x| \geq (\sqrt{10} - 1)R^{1/2}$$

for all $s \in [0, 1]$, so that

$$\inf_{s \in [0, 1]} \frac{|x + s(y - x)|^2}{R} \geq (\sqrt{10} - 1)^2 > 3,$$

and the desired conclusion follows from the hypothesis $\text{supp } \rho \subset (1, 3)$.

It follows that whenever

$$\rho_*(x, y) := \sup_{s \in [0, 1]} \rho \left(\frac{|x + s(y - x)|^2}{R} \right)$$

is nonzero, we have

$$\frac{|x + s(y - x)|^2}{R} \leq \frac{2(|x|^2 + |y - x|^2)}{R} \leq 22.$$

Choosing $c_0(\rho)$ as in (2.11), we therefore obtain

$$\sup_s \frac{|x + s(y - x)|^2}{R} \rho \left(\frac{|x + s(y - x)|^2}{R} \right) \leq 22 \|\rho\|_{L^\infty} \leq \frac{22}{c_0(\rho)} F_R(|y|).$$

This again leads to the desired estimate.

Since these three cases cover all possible values of $|x|$ and $|y|$, this completes the proof of the lemma. \square

3. VIRIAL AND LOCALIZED VIRIAL IDENTITIES

In this section, we establish a virial identity for (1.1) and obtain localized variants of this result. The general procedure for this derivation follows a classical approach; for a particular case of the analysis which is particularly relevant to hierarchies of the form (1.1), see the related results in [4]. We begin with a preliminary lemma.

Lemma 3.1. *Let $(\gamma^{(k)})_{k \geq 1}$ be an $(A)-(D)$ -admissible solution to (1.1) and define $P : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ by*

$$P(t, x) := \int e^{ix \cdot (\xi - \xi')} (\xi + \xi') \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' \quad (3.1)$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. We then have

$$\partial_t \gamma^{(1)}(t, x, x) + \operatorname{div}_x P(t, x) = 0$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$.

Proof. Proceeding by direct computation, note that after taking the Fourier transform and its inverse (and using that $(\gamma^{(k)})$ solves (1.1)) we obtain

$$\begin{aligned} \partial_t \gamma^{(1)}(x, x) &= \int e^{ix \cdot (\xi - \xi')} \partial_t \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' \\ &= i \int e^{ix \cdot (\xi - \xi')} (|\xi'|^2 - |\xi|^2) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' - \mu i (B_{1,2}^\pm \gamma^{(2)})(x, x). \end{aligned} \quad (3.2)$$

Now, writing $\partial_{x_i} e^{ix \cdot (\xi - \xi')} = (\xi_i - \xi'_i) e^{ix \cdot (\xi - \xi')}$ for $1 \leq i \leq d$, we get

$$\begin{aligned} (3.2) &= -\operatorname{div}_x \int e^{ix \cdot (\xi - \xi')} (\xi' + \xi) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' \\ &\quad - \mu i \int \gamma^{(2)}(x, y, x', y) (V(x - y) - V(x' - y)) \Big|_{x'=x} dy \\ &= -\operatorname{div}_x P, \end{aligned}$$

as desired. \square

With this lemma in hand, we next establish the appropriate virial identity for (1.1).

Proposition 3.2 (Virial identity for (1.1)). *Let $(\gamma^{(k)})_{k \geq 1}$ be an $(A)-(D)$ -admissible solution to (1.1). Then we have the identity*

$$\partial_{tt} \operatorname{Tr} (|x|^2 \gamma^{(1)}) = 8 \operatorname{Tr} (-\Delta_{x_1} \gamma^{(1)}) - 4\mu \int \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy.$$

Proof. Invoking Lemma 3.1 and integrating by parts, we write

$$\begin{aligned} \partial_{tt} \operatorname{Tr} (|x|^2 \gamma^{(1)}) &= \int |x|^2 \partial_{tt} \gamma(x, x) dx \\ &= - \int |x|^2 \operatorname{div}_x \partial_t P dx \\ &= 2 \int x \cdot \partial_t P dx. \end{aligned}$$

Next, expanding the definition of $P(t, x)$ and using that $(\gamma^{(k)})_{k \geq 1}$ solve (1.1), this last expression becomes

$$\begin{aligned} &2 \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') \partial_t \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &= 2i \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') (|\xi'|^2 - |\xi|^2) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &\quad - 2\mu i \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') \widehat{B_{1,2}^\pm \gamma^{(2)}}(\xi, \xi') d\xi d\xi' dx. \end{aligned} \quad (3.3)$$

To simplify the first term, we use an argument from [4] (see in particular Section 5.2 of [4]). This computation proceeds as follows: first, letting $x \otimes y$ denote the operator defined by $(x \otimes y)z = (y \cdot z)x$ for $x, y, z \in \mathbb{R}^d$, integration by parts and Fubini's theorem yield

$$\begin{aligned}
& 2i \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') (|\xi'|^2 - |\xi|^2) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\
&= -2i \int e^{ix \cdot (\xi - \xi')} x \cdot [(\xi + \xi') \otimes (\xi + \xi')] (\xi - \xi') \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\
&= 2 \int e^{ix \cdot (\xi - \xi')} \text{tr} [(\xi + \xi') \otimes (\xi + \xi')] \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\
&= 8 \int |\xi|^2 \widehat{\gamma^{(1)}}(\xi, \xi) d\xi
\end{aligned} \tag{3.4}$$

where $\text{tr}(A) = \sum_{i=1}^d (Ae_i) \cdot e_i$ is the usual matrix trace operator. Now, using the assumption that $(\gamma^{(k)})_{k \geq 1}$ is Hermitian (since it is (A)-(D)-admissible), we may appeal to Parseval's identity to obtain

$$(3.4) = 8 \sum_{j=1}^{\infty} \int \lambda_j^2 |(i\xi) \widehat{g_j}(\xi)|^2 d\xi = 8 \sum_{j=1}^{\infty} \int \lambda_j^2 |\nabla g_j(x)|^2 dx = 8 \text{Tr}(-\Delta_{x_1} \gamma^{(1)}),$$

where (λ_j) and (g_j) are suitably chosen sequences.

Substituting these identities back into (3.3), we obtain

$$(3.3) = 8 \text{Tr}(-\Delta_{x_1} \gamma^{(1)}) + (II),$$

where

$$(II) := -2\mu i \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') \widehat{B_{1,2}^{\pm} \gamma^{(2)}}(\xi, \xi') d\xi d\xi' dx.$$

To complete the proof of the proposition, it remains to show

$$(II) = -4\mu \int \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy. \tag{3.5}$$

To accomplish this, we note that a direct calculation allows us to compute the Fourier transform of $B_{1,2}^+$ acting on $\gamma^{(2)}$ as

$$\begin{aligned}
\widehat{B_{1,2}^+ \gamma^{(2)}}(\xi, \xi') &= \int [\widehat{\gamma^{(2)}}(\xi - q + q', q, \xi', q') \\
&\quad - \widehat{\gamma^{(2)}}(\xi, q, \xi' + q - q', q')] \widehat{V}(q - q') dq dq'.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
(II) &= -2\mu i \int e^{ix \cdot (\xi - \xi')} x \cdot (\xi + \xi') [\widehat{\gamma^{(2)}}(\xi - q + q', q, \xi', q') \\
&\quad - \widehat{\gamma^{(2)}}(\xi, q, \xi' + q - q', q')] \widehat{V}(q - q') dq dq' d\xi d\xi' dx
\end{aligned}$$

which, in view of the changes of variables $\xi \mapsto \xi - q + q'$ and $\xi' \mapsto \xi' - q + q'$ in the first and second terms, respectively, is equal to

$$-2\mu i \int e^{ix \cdot (\xi - \xi')} x \cdot (2q - 2q') \widehat{\gamma^{(2)}}(\xi - q + q', q, \xi', q') \widehat{V}(q - q') dq dq' d\xi d\xi' dx. \tag{3.6}$$

Expanding $\widehat{\gamma^{(2)}}$ and \widehat{V} in this expression by the definition of the Fourier transform, we obtain

$$(3.6) = -4\mu i \int e^{i\Phi} x \cdot (q - q') \gamma^{(2)}(z, y, z', y') V(w) d\Lambda,$$

where $\Lambda = (x, w, y, z, q, \xi, y', z', q', \xi')$ and with $\Phi = \Phi(\Lambda)$ defined by

$$\Phi(\Lambda) := (x - z) \cdot \xi - (x - z') \cdot \xi' + (z - y - w) \cdot q - (z - y' - w) \cdot q'.$$

By the Fubini theorem, a distributional calculation (first in the variables ξ and z , resulting in restriction to the set where $z = x$; then in the variables ξ' and z' , resulting in restriction to the set where $z' = x$) shows that this is equal to

$$-4\mu \int x \cdot \nabla_x [e^{i\tilde{\Phi}}] \gamma^{(2)}(x, y, x, y') V(w) d\tilde{\Lambda}$$

with $\tilde{\Lambda} = (x, w, y, q, y', q')$ and $\tilde{\Phi} = \tilde{\Phi}(\tilde{\Lambda}) = (x - y - w) \cdot q - (x - y' - w) \cdot q'$. Integrating by parts, this becomes

$$\begin{aligned} & 4\mu d \int e^{i\tilde{\Phi}} \gamma^{(2)}(x, y, x, y') V(w) d\tilde{\Lambda} \\ & + 4\mu \int e^{i\tilde{\Phi}} x \cdot \nabla_x [\gamma^{(2)}(x, y, x, y')] V(w) d\tilde{\Lambda} \end{aligned}$$

Evaluating the integrals in q', y', q and w , and recalling that $(\gamma^{(k)})$ is (A)–(D)-admissible (and thus in particular satisfies (D)), the above expression becomes

$$4\mu d \int B_{1,2}^+ \gamma^{(2)}(x, x) dx + 4\mu \int x \cdot \nabla_x [\gamma^{(2)}(x, y, x, y)] V(x - y) dx dy$$

which, after another application of integration by parts, is equal to the desired quantity in (3.5). This completes the proof of the proposition. \square

Our next result is a localized version of Proposition 3.2, in which the weight $|x|^2$ is replaced with an arbitrary smooth cutoff function.

Proposition 3.3 (Localized virial identity for (1.1)). *Fix $\phi \in C_c^\infty(\mathbb{R}^d)$ and let $(\gamma^{(k)})_{k \geq 1}$ be an (A)–(D)-admissible solution to (1.1). Then we have the identity*

$$\begin{aligned} \partial_{tt} \text{Tr}(\phi \gamma^{(1)}) &= 2 \text{Re} \int H_x(\phi)(x) \cdot \left(H_{x,x'}(\gamma^{(1)})(x, x) - H_{x,x}(\gamma^{(1)})(x, x) \right) dx \\ &\quad - 2\mu \int \gamma^{(2)}(x, y, x, y) (\nabla \phi)(x) \cdot (\nabla V)(x - y) dx dy \end{aligned}$$

where $H_x(\phi)$, $H_{x,x}(f)$ and $H_{x,x'}(f)$ (with $f : I \times \mathbb{R}^{dk} \times \mathbb{R}^{dk} \rightarrow \mathbb{C}$) are the $d \times d$ matrices $H_x(\phi) = (\partial_{x_i} \partial_{x_j} \phi)_{i,j}$, $H_{x,x}(f) = (\partial_{x_i} \partial_{x_j} f)_{i,j}$ and $H_{x,x'}(f) = (\partial_{x_i} \partial_{x'_j} f)_{i,j}$.

Before beginning the proof of Proposition 3.3, we recall that, as a basic consequence of the Hermitian property of (A)–(D)-admissible solutions, for $1 \leq i \leq d$ we have the identities

$$(\partial_{x_i} \gamma^{(1)})(x, x) = \overline{(\partial_{x'_i} \gamma^{(1)})(x, x)}, \quad (3.7)$$

$$(\partial_{x_i} \partial_{x_j} \gamma^{(1)})(x, x) = \overline{(\partial_{x'_i} \partial_{x'_j} \gamma^{(1)})(x, x)}, \quad (3.8)$$

and

$$(\partial_{x_i} \partial_{x'_j} \gamma^{(1)})(x, x) = \overline{(\partial_{x'_i} \partial_{x_j} \gamma^{(1)})(x, x)}, \quad (3.9)$$

for $x \in \mathbb{R}^d$. These identities will play a key role in the analysis which follows.

Proof of Proposition 3.3. As in the proof of Proposition 3.2, we use integration by parts to write

$$\partial_{tt} \operatorname{Tr}(\phi \gamma^{(1)}) = \int \nabla_x \phi(x) \cdot \partial_t P dx, \quad (3.10)$$

and use the definition of $P(t, x)$ to obtain

$$\begin{aligned} (3.10) &= \int e^{ix \cdot (\xi - \xi')} \nabla \phi(x) \cdot (\xi + \xi') \partial_t \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &= \int i e^{ix \cdot (\xi - \xi')} \nabla \phi(x) \cdot (\xi + \xi') (|\xi'|^2 - |\xi|^2) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &\quad - \mu i \int e^{ix \cdot (\xi - \xi')} \nabla \phi(x) \cdot (\xi + \xi') \widehat{B_{1,2}^\pm \gamma^{(2)}}(\xi, \xi') d\xi d\xi' dx. \end{aligned} \quad (3.11)$$

Performing a similar calculation as before on the first term, we obtain

$$\begin{aligned} &\int i e^{ix \cdot (\xi - \xi')} \nabla \phi(x) \cdot (\xi + \xi') (|\xi'|^2 - |\xi|^2) \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &= \sum_{j,k=1}^d \int (\partial_{x_k} \partial_{x_j} \phi)(x) e^{ix \cdot (\xi - \xi')} (\xi + \xi')_k (\xi + \xi')_j \widehat{\gamma^{(1)}}(\xi, \xi') d\xi d\xi' dx \\ &= \sum_{j,k} \int (\partial_{x_k} \partial_{x_j} \phi)(x) \left(-(\partial_{x_k} \partial_{x_j} \gamma^{(1)})(x, x) + (\partial_{x_k} \partial_{x'_j} \gamma^{(1)})(x, x) \right. \\ &\quad \left. + (\partial_{x'_k} \partial_{x_j} \gamma^{(1)})(x, x) - (\partial_{x'_k} \partial_{x'_j} \gamma^{(1)})(x, x) \right) dx. \end{aligned}$$

In view of this, applying (3.8) and (3.9) and defining (II_ϕ) by

$$(II_\phi) := -\mu i \int e^{ix \cdot (\xi - \xi')} \nabla \phi(x) \cdot (\xi + \xi') \widehat{B_{1,2}^\pm \gamma^{(2)}}(\xi, \xi') d\xi d\xi' dx,$$

we obtain that the right side of (3.11) is equal to

$$2\operatorname{Re} \int H_x(\phi)(x) \cdot \left(H_{x,x'}(\gamma^{(1)})(x, x) - H_{x,x}(\gamma^{(1)})(x, x) \right) dx + (II_\phi).$$

It remains to evaluate (II_ϕ) . As in Proposition 3.2, this is accomplished by a simple distributional calculation: computing $\widehat{B_{1,2}^\pm}$, expanding $\widehat{\gamma^{(2)}}$ and \widehat{V} via the definition of the Fourier transform, and applying the Fubini theorem. Indeed, this procedure of calculation gives

$$(II_\phi) = -2\mu \int \gamma^{(2)}(x, y, x, y) (\nabla \phi)(x) \cdot (\nabla V)(x - y) dx dy,$$

exactly as in the second half of the proof of Proposition 3.2. This completes the proof of the proposition. \square

To conclude this section, we include a brief lemma showing how the first term on the right side of the identity in Proposition 3.3 can be evaluated further, using the Hermitian property of (A)–(D)-admissible solutions.

Lemma 3.4. Fix $\phi \in C_c^\infty$, and let $(\gamma^{(k)})_{k \geq 1}$ be an $(A)-(D)$ -admissible solution to (1.1). Then

$$\begin{aligned} & 2\operatorname{Re} \int H_x(\phi) \cdot H_{x,x}(\gamma^{(1)})(x, x) dx \\ &= \int \Delta^2(\phi)(x) \gamma^{(1)}(x, x) dx - 2\operatorname{Re} \int H_x(\phi) \cdot H_{x',x}(\gamma^{(1)})(x, x) dx, \end{aligned}$$

where $H_x(\phi)$, $H_{x,x}(f)$ and $H_{x',x}(f)$ are as defined in the statement of Lemma 3.3.

Proof. Since $(\gamma^{(k)})$ is Hermitian and $\gamma^{(k)} \geq 0$, we may find sequences (λ_ℓ) and (g_ℓ) such that

$$\gamma^{(1)}(x, x') = \sum_{\ell=1}^{\infty} \lambda_\ell g_\ell(x) \overline{g_\ell(x')}$$

for all $(x, x') \in \mathbb{R}^{2d}$. We then integrate by parts (writing $\partial_{j,k} = \partial_{x_j} \partial_{x_k}$ and $\partial_{j,j,k} = \partial_{x_j}^2 \partial_{x_k}$) to obtain

$$\begin{aligned} & \int H_x(\phi)(x) \cdot H_{x,x}(\gamma^{(1)})(x, x) dx \\ &= \sum_{j,k,\ell=1}^d \lambda_\ell \int (\partial_{j,k}\phi)(x) (\partial_{j,k}g_\ell)(x) \overline{g_\ell(x)} dx \\ &= - \sum_{j,k,\ell} \lambda_\ell \int \left[(\partial_{j,j,k}\phi)(x) \overline{g_\ell(x)} + (\partial_{j,k}\phi)(x) \overline{(\partial_j g_\ell)(x)} \right] (\partial_{x_k} g_\ell)(x) dx \end{aligned}$$

A second integration by parts shows that this is equal to

$$\begin{aligned} & \sum_{j,k,\ell=1}^d \lambda_\ell \int \left((\partial_{j,j,k}\phi) \overline{g_\ell(x)} + (\partial_{j,j,k}\phi) \overline{\partial_k g_\ell(x)} \right) g_\ell(x) dx \\ & \quad - \sum_{j,k,\ell=1}^d \lambda_\ell \int (\partial_{j,k}\phi) \overline{(\partial_j g_\ell)(x)} (\partial_k g_\ell)(x) dx \end{aligned} \quad (3.12)$$

A final integration by parts now gives

$$\begin{aligned} (3.12) &= \int \Delta^2(\phi)(x) \gamma^{(1)}(x, x) dx - \sum_{j,k,\ell} \lambda_\ell \int (\partial_{j,k}\phi)(x) \partial_j [\overline{\partial_k g_\ell(x)} g_\ell(x)] dx \\ & \quad - \sum_{j,k,\ell} \lambda_\ell \int (\partial_{j,k}\phi)(x) \overline{(\partial_j g_\ell)(x)} (\partial_k g_\ell)(x) dx \\ &= \int \Delta^2(\phi)(x) \gamma^{(1)}(x, x) dx - \int H_x(\phi) \cdot H_{x,x}(\gamma^{(1)})(x, x) dx \\ & \quad - 2\operatorname{Re} \sum_{j,k,\ell} \lambda_\ell \int (\partial_{j,k}\phi)(x) \overline{(\partial_j g_\ell)(x)} (\partial_k g_\ell)(x) dx, \end{aligned}$$

which yields

$$\begin{aligned} 2\operatorname{Re} \int H_x(\phi) \cdot H_{x,x}(\gamma^{(1)})(x, x) dx &= \int \Delta^2(\phi)(x) \gamma^{(1)}(x, x) dx \\ & \quad - 2\operatorname{Re} \int H_x(\phi) \cdot H_{x',x}(\gamma^{(1)})(x, x) dx \end{aligned}$$

as desired. \square

4. PROOFS OF THE MAIN THEOREMS: NEGATIVE ENERGY BLOW-UP SOLUTIONS

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2, our results on negative energy finite-time blowup for (1.1). We begin with the proof of Theorem 1.1, for which our arguments are in the spirit of the classical Glassey argument (see also [4] for a similar application of this argument to negative-energy blowup for the Gross-Pitaevskii hierarchy).

Proof of Theorem 1.1. Define $V_1(t) = \text{Tr}(|x|^2 \gamma^{(1)}(t))$. Since $(\gamma^{(k)})$ is Hermitian with $\gamma^{(k)} \succeq 0$, we may find $\lambda_j \geq 0$ and $\psi_j : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$V_1(t) = \sum_{j=1}^{\infty} \int |x|^2 \lambda_j |\psi_j(x)|^2 dx \geq 0.$$

Suppose for contradiction that the claim fails. Then $(\gamma^{(k)})_{k \geq 1}$ is a global solution, and thus $V_1(t)$ is defined for all $t \in \mathbb{R}$ with $V(t) \geq 0$ everywhere. Recall that Proposition 3.2 implies the bound

$$\begin{aligned} \partial_{tt} V_1(t) &= 8 \text{Tr}(-\Delta_{x_1} \gamma^{(1)}) + 4 \int \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dy dx \\ &= 16E_1(t) + 4 \int \gamma^{(2)}(x, y, x, y) V(x - y) dy dx \\ &\quad + 4 \int \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dy dx \\ &= 16E_1(t) + 4 \int \gamma^{(2)}(x, y, x, y) V(x - y) dy dx \\ &\quad + 2 \int \gamma^{(2)}(x, y, x, y) (x - y) \cdot (\nabla V)(x - y) dy dx \\ &= 16E_1(t) + 4[(V + \frac{1}{2} x \cdot (\nabla V)) * \gamma^{(2)}(x, \cdot, x, \cdot)](x) \\ &\leq 16E_1(t) \end{aligned}$$

where we have used the identity

$$\begin{aligned} &\int \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dy dx \\ &= - \int \gamma^{(2)}(x, y, x, y) y \cdot (\nabla V)(x - y) dy dx, \end{aligned} \tag{4.1}$$

which is a consequence of the symmetry property of $(\gamma^{(k)})$ and the assumption that V is even.

However, $t \mapsto E_1(t)$ is constant by Proposition 2.2, the conservation of energy. In particular, we obtain $V_1(t) < 0$ for $|t|$ is sufficiently large, which contradicts the positivity of V_1 . Thus, we conclude that $(\gamma^{(k)})$ cannot be globally defined. \square

We now turn to the proof of Theorem 1.2. For these arguments, we make use of the truncated virial identity established in Proposition 3.3. We begin with a variant of this proposition which is adapted to an assumption of radial symmetry and to the particular rescaled cutoff we wish to use.

Lemma 4.1. *Set $\mu = -1$. Fix $\psi \in C_0^\infty([0, \infty))$ and for each $R > 0$ let ψ_R be given by*

$$\psi_R(x) = R\psi\left(\frac{|x|^2}{R}\right) \quad \text{for } x \in \mathbb{R}^d.$$

Suppose that $(\gamma^{(k)})_{k \geq 1}$ is an (A)-(D)-admissible solution to (1.1) which is radially symmetric in the x and x' variables respectively (in the sense given in the statement of Theorem 1.2). We then have

$$\begin{aligned} & \partial_{tt} \operatorname{Tr}(\psi_R \gamma^{(1)}) \\ & \leq 16E_1 - 8 \int (1 - \psi'(\frac{|x|^2}{R}) - 2|x|^2 R^{-1} \psi''(\frac{|x|^2}{R})) (\partial_{r,r'} \gamma^{(1)})(x, x) dx \\ & \quad + 4 \int (V(x - y) + \frac{1}{2}(x - y) \cdot (\nabla V)(x - y)) \gamma^{(2)}(x, y, x, y) dx dy \\ & \quad - 2 \int a(x, y) \cdot (\nabla V)(x - y) \gamma^{(2)}(x, y, x, y) dx dy \\ & \quad - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx \end{aligned} \quad (4.2)$$

for every $R > 0$, where we have written $r = |x|$, $r' = |x'|$, and set

$$a(x, y) := (x - y) - (\psi'(\frac{|x|^2}{R})x - \psi'(\frac{|y|^2}{R})y), \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (4.3)$$

Proof. Let $R > 0$ be given. Applying Proposition 3.3 and Lemma 3.4, we obtain (since $\mu = -1$)

$$\begin{aligned} & \partial_{tt} \operatorname{Tr}(\psi_R \gamma^{(1)}) \\ & = 4\operatorname{Re} \int H_x(\psi_R)(x) \cdot H_{x,x'}(\gamma^{(1)})(x, x) dx - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx \\ & \quad + 2 \int \gamma^{(2)}(x, y, x, y) (\nabla \psi_R)(x) \cdot (\nabla V)(x - y) dx dy. \end{aligned}$$

Setting $r = |x|$, $r' = |x'|$, and computing derivatives of ψ_R , this is equal to

$$\begin{aligned} & 4\operatorname{Re} \sum_{j,k} \int \partial_{x_j} [2x_k \psi'(\frac{|x|^2}{R})] (\partial_{x_j} [\partial_{r'} \gamma^{(1)} \frac{x'_k}{|x'|}]) \Big|_{(x,x)} - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx \\ & \quad + 4 \int \psi'(\frac{|x|^2}{R})(x) \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy \end{aligned}$$

which, integrating by parts, becomes

$$\begin{aligned} & 8\operatorname{Re} \sum_j \int \left[\frac{x_j^2}{|x|^2} \psi'(\frac{|x|^2}{R}) (\partial_{rr'} \gamma^{(1)})(x, x) \right. \\ & \quad \left. + \sum_k \frac{2x_k^2 x_j^2}{R|x|^2} \psi''(\frac{|x|^2}{R}) (\partial_{rr'} \gamma^{(1)})(x, x) \right] dx \\ & \quad - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx \\ & \quad + 4 \int \psi'(\frac{|x|^2}{R})(x) \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy. \end{aligned} \quad (4.4)$$

Now, evaluating the sum in j and k , and observing that since $(\gamma^{(k)})$ is (A)–(D)-admissible (and in particular Hermitian), $\partial_{r,r'}\gamma^{(1)}(x, x)$ is real for all $x \in \mathbb{R}^d$, we obtain

$$(4.4) = 8 \int \left[\psi' \left(\frac{|x|^2}{R} \right) (\partial_{r,r'}\gamma^{(1)})(x, x) dx + \frac{2|x|^2}{R} \psi'' \left(\frac{|x|^2}{R} \right) (\partial_{r,r'}\gamma^{(1)})(x, x) \right] dx \\ + 4 \int \psi' \left(\frac{|x|^2}{R} \right) (x) \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy \\ - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx, \quad (4.5)$$

We next add and subtract $16E_1$, where E_1 is as the energy defined in (1.4), which is conserved in time by Proposition 2.2. This gives

$$(4.5) = 16E_1 - 8 \int \left(1 - \psi' \left(\frac{|x|^2}{R} \right) - \frac{2|x|^2}{R} \psi'' \left(\frac{|x|^2}{R} \right) \right) (\partial_{r,r'}\gamma^{(1)})(x, x) dx \\ + 4 \int \gamma^{(2)}(x, y, x, y) (V(x - y) + \frac{1}{2}(x - y) \cdot (\nabla V)(x - y)) dx dy \\ - 2 \int \gamma^{(2)}(x, y, x, y) (x - y) \cdot (\nabla V)(x - y) dx dy \\ + 4 \int \psi' \left(\frac{|x|^2}{R} \right) \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dx dy \\ - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx.$$

The desired result now follows from the identity,

$$\int \psi' \left(\frac{|x|^2}{R} \right) \gamma^{(2)}(x, y, x, y) x \cdot (\nabla V)(x - y) dy dx \\ = - \int \psi' \left(\frac{|y|^2}{R} \right) \gamma^{(2)}(x, y, x, y) y \cdot (\nabla V)(x - y) dy dx,$$

which, in a similar manner to the related identity (4.1) used in the proof of Theorem 1.1, follows from the symmetry properties of (A)–(D)-admissible solutions. \square

We now continue with the proof of Theorem 1.2. We remark that the general pattern of argument is closely related to the approach of Ogawa and Tsutsumi [15]; see also [10] for an earlier application of the method to the nonlinear Schrödinger equation with Hartree nonlinearity, as well as a textbook treatment in [1, Theorem 6.5.10].

In the proof, we will make essential use of the quantum de Finetti theorem of Hudson and Moody [11] (see [12] as well as [3] for the particular formulation applicable to $\gamma^{(2)}$); applying this theorem with the base Hilbert space taken as $H = L^2_{\text{rad}} \subset L^2(\mathbb{R}^d)$, the subspace of L^2 consisting of radial functions, this gives the existence of a Borel measure μ on L^2_{rad} , supported on $\{f \in L^2_{\text{rad}} : \|f\|_{L^2} = 1\}$, with $\mu(L^2_{\text{rad}}) = 1$, and such that both

$$\gamma^{(1)}(x, x) = \int |\phi(x)|^2 d\mu(\phi) \quad (4.6)$$

and

$$\gamma^{(2)}(x, x, y, y) = \int |\phi(x)|^2 |\phi(y)|^2 d\mu(\phi) \quad (4.7)$$

hold in the sense of distributions.

We will also several times use the observation that $(\gamma^{(k)}) \succeq 0$ and $(\gamma^{(k)})$ Hermitian together imply

$$\gamma^{(2)}(x, y, x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^d. \quad (4.8)$$

Proof of Theorem 1.2. For each $R > 0$, define ψ_R by $\psi_R(x) = R\psi(\frac{|x|^2}{R})$ as in Lemma 4.1. Applying Lemma 4.1 and using (1.5) and (4.8) to see that the third term of the resulting bound is non-positive, we bound $\partial_{tt} \text{Tr}(\psi_R \gamma^{(1)})$ by

$$16E_1 + (II) + (III) + (IV), \quad (4.9)$$

with

$$(II) := -8 \int \left(1 - \psi'(\frac{|x|^2}{R}) - 2|x|^2 R^{-1} \psi''(\frac{|x|^2}{R}) \right) (\partial_{r,r'} \gamma^{(1)})(x, x) dx,$$

$$(III) := -2 \int a(x, y) \cdot (\nabla V)(x - y) \gamma^{(2)}(x, y, x, y) dx dy,$$

and

$$(IV) := - \int \Delta^2(\psi_R)(x) \gamma^{(1)}(x, x) dx.$$

As in the proof of Theorem 1.1, our goal is to bound this by a negative quantity, uniformly in t . Since $E_1 < 0$ by assumption, it suffices to estimate the sum of (II), (III), and (IV).

We begin by re-expressing (II) in a more convenient form (recognizing it as a strictly negative quantity). Note that

$$1 - \psi'(\frac{|x|^2}{R}) - 2|x|^2 R^{-1} \psi''(\frac{|x|^2}{R}) = 2|x|^2 R^{-1} \rho\left(\frac{|x|^2}{R}\right) + \int_0^{|x|^2 R^{-1}} \rho(y) dy$$

for every $x \in \mathbb{R}^d$. Combined with the integral expression (4.6) for $\gamma^{(1)}$ (with respect to the measure μ on L_{rad}^2), this leads (via an application of the Tonelli theorem to interchange the order of integration) to the representation

$$\begin{aligned} (II) &= -8 \int \left(F_R(|x|) + \frac{2|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) (\partial_{r,r'} \gamma^{(1)})(x, x) dx \\ &= -8 \int \left(F_R(|x|) + \frac{2|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) |\nabla \phi(x)|^2 dx d\mu(\phi). \end{aligned} \quad (4.10)$$

We now turn to (III). Observing that $a(x, y) = 0$ on $(\mathbb{R}^d \times \mathbb{R}^d) \setminus A_R$, with $A_R := \{(x, y) : \max\{|x|, |y|\} \geq R^{1/2}\}$, this term becomes

$$\begin{aligned} &-2 \int_{A_R} a(x, y) \cdot (\nabla V)(x - y) \gamma^{(2)}(x, y, x, y) dx dy \\ &\leq 2 \int_{A_R \cap \{(x, y) : |x - y| > R^{1/2}\}} |a(x, y) \cdot (\nabla V)(x - y)| \gamma^{(2)}(x, y, x, y) dx dy \\ &\quad + 2 \int_{A_R \cap \{(x, y) : |x - y| \leq R^{1/2}\}} |a(x, y) \cdot (\nabla V)(x - y)| \gamma^{(2)}(x, y, x, y) dx dy. \end{aligned} \quad (4.11)$$

where we have used (4.8). We let the first and second terms in (4.11) be denoted by (IIIa) and (IIIb), respectively.

To estimate (IIIa), note that we may find $C > 0$ such that $|a(x, y)| \leq C|x - y|$ for all x and y . We then have

$$\begin{aligned} (IIIa) &\leq C \int_{|x-y| \geq R^{1/2}} |x-y| |(\nabla V)(x-y)| \gamma^{(2)}(x, y, x, y) dx dy \\ &\leq C \sup_{|x-y| \geq R^{1/2}} \left(|x-y| |(\nabla V)(x-y)| \right) \int \gamma^{(2)}(x, y, x, y) dx dy \\ &= C \sup_{|x-y| \geq R^{1/2}} \left(|x-y| |(\nabla V)(x-y)| \right) \int \gamma_0^{(1)}(x, x) dx. \end{aligned}$$

where we have used the admissibility of $\gamma^{(1)}$ and Proposition 2.1 to obtain the last equality. We may then use the hypothesis (1.6) to choose R sufficiently large so that

$$(IIIa) \leq 4|E_1|. \quad (4.12)$$

We now turn to (IIIb). Applying Lemma 2.4, we obtain

$$\begin{aligned} (IIIb) &\leq 2C \int_{A_R \cap \{(x, y): |x-y| \leq R^{1/2}\}} \left(F_R(|x|) + F_R(|y|) \right. \\ &\quad \left. + \frac{|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) + \frac{|y|^2}{R} \rho\left(\frac{|y|^2}{R}\right) \right) |x-y| \\ &\quad \cdot |(\nabla V)(x-y)| \gamma^{(2)}(x, y, x, y) dx dy, \end{aligned}$$

which, in view of the symmetry properties of (A)–(D)-admissible solutions, is bounded by a multiple of

$$\begin{aligned} &\int_{A_R \cap \{(x, y): |x-y| \leq R^{1/2}\}} \left(F_R(|x|) + \frac{|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) |x-y| \\ &\quad \cdot |(\nabla V)(x-y)| \gamma^{(2)}(x, y, x, y) dx dy. \end{aligned} \quad (4.13)$$

We now make use of (4.7), the integral representation for $\gamma^{(2)}$ given by the quantum de Finetti theorem. Substituting this into our bound for (IIIb), we get (in view of the Tonelli theorem, and the Hölder and Young inequalities)

$$\begin{aligned} (4.13) &\lesssim \int \left(F_R(|x|) + \frac{|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) |\phi(x)|^2 \left((|x| |\nabla V(x)| \chi_R(x)) * |\phi|^2 \right)(x) dx d\mu(\phi) \\ &\leq \int \left\| \left(F_R(|x|) + \frac{|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) |\phi(x)|^2 \right\|_{L_x^\infty} \\ &\quad \cdot \left\| (|x| |\nabla V(x)| \chi_R(x)) * |\phi(x)|^2 \right\|_{L_x^1} d\mu(\phi) \\ &\lesssim \int \left\| \left(F_R(|x|) + \frac{|x|^2}{R} \rho\left(\frac{|x|^2}{R}\right) \right) |\phi(x)|^2 \right\|_{L_x^\infty} \\ &\quad \cdot \| |x| |\nabla V(x)| \|_{L_x^1(|x| \leq R^{1/2})} \|\phi\|_{L_x^2}^2 d\mu(\phi), \end{aligned}$$

where we have let $\chi_R = \chi_{\{|x| \leq R^{1/2}\}}$ denote the characteristic function of the ball of radius $R^{1/2}$ centered at the origin.

Now, note that $F_R(|x|) = 0$ and $\rho(|x|^2/R) = 0$ for $|x|^2 \leq R$, and, moreover, there exists $C > 0$ such that $F_R(|x|) \leq C$ and $\frac{|x|^2}{R} \rho(|x|^2/R) \leq C$ for all $x \in \mathbb{R}^d$.

Combining these bounds with the equality

$$\|\phi\|_{L^2} = 1, \quad (4.14)$$

which is valid on the support of μ , the above expression is bounded by a multiple of

$$A_R \int \left\| \left(F_R(|x|) + \frac{|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) \right)^{1/2} \phi \right\|_{L_x^\infty(|x| \geq R^{1/2})}^2 d\mu(\phi), \quad (4.15)$$

with $A_R := \| |x| |\nabla V(x)| \|_{L_x^1(|x| \leq R^{1/2})}$.

Recalling that μ is a measure on L_{rad}^2 , we now invoke a form of the Strauss lemma for radial functions (see, e.g. Lemma 1.7.3 and Lemma 6.5.11 in [1]), giving the inequality

$$\begin{aligned} \| |x|^{(N-1)/2} f(x) g(x) \|_{L_x^\infty}^2 &\lesssim \| f \nabla f \|_{L_x^\infty} \| g \|_{L_x^2}^2 + \| f g \|_{L_x^2} \| f \nabla g \|_{L_x^2} \\ &\lesssim \| f \nabla f \|_{L_x^\infty} \| g \|_{L_x^2}^2 + \| f g \|_{L_x^2}^2 + \| f \nabla g \|_{L_x^2}^2 \end{aligned}$$

with $f \in C^1(\mathbb{R}^N)$ and $g \in H^1(\mathbb{R}^N)$ both radial, where we have fixed $N \geq 2$. Applying this inequality with $f(x) = (F_R(|x|) + \frac{|x|^2}{R} \rho(\frac{|x|^2}{R}))^{1/2}$, $x \in \mathbb{R}^N$ and $g = \phi$, we compute

$$|(f \nabla f)(x)| \lesssim \left(\frac{|x|}{R} + \frac{|x|^3}{R^2} \right) \|\rho\|_{C^1(\mathbb{R})} \chi_{\{x: 1 < |x|^2/R < 3\}}(x) \lesssim R^{-1/2}$$

for $x \in \mathbb{R}^N$, and therefore obtain

$$\begin{aligned} (4.15) &\lesssim \frac{A_R}{R^{(N-1)/2}} \int \left(R^{-1/2} \|\phi\|_{L_x^2}^2 + \|\phi\|_{L^2}^2 \right. \\ &\quad \left. + \left\| \left(F_R(|x|) + \frac{|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) \right)^{1/2} |\nabla \phi| \right\|_{L_x^2}^2 \right) d\mu(\phi) \\ &\lesssim \frac{A_R}{R^{(N-1)/2}} \int \left(1 + \left\| \left(F_R(|x|) + \frac{2|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) \right)^{1/2} |\nabla \phi| \right\|_{L_x^2}^2 \right) d\mu(\phi), \end{aligned} \quad (4.16)$$

where we have used the observations made above about boundedness and support of $x \mapsto F_R(|x|)$ and $x \mapsto \frac{|x|^2}{R} \rho(|x|^2/R)$, and again invoked (4.14).

Rewriting the right side of (4.16) as

$$\begin{aligned} &\frac{A_R}{R^{(N-1)/2}} + \frac{A_R}{R^{(N-1)/2}} \int \left(F_R(|x|) + \frac{2|x|^2}{R} \rho \left(\frac{|x|^2}{R} \right) \right) |\nabla \phi|^2 dx d\mu(\phi) \\ &= \frac{A_R}{R^{(N-1)/2}} + \frac{A_R}{8R^{(N-1)/2}} |(II)|, \end{aligned}$$

where the last equality follows from (4.10), the hypothesis (1.7) implies that for R sufficiently large (to ensure $CA_R/R^{(N-1)/2} \leq 4|E_1|$ and $CA_R/(8R^{(N-1)/2}) \leq 1/2$) we have the bound

$$(IIIb) \leq 4|E_1| + \frac{1}{2} |(II)|. \quad (4.17)$$

We now estimate (IV). Note that $\Delta^2 \psi_R = 0$ for $|x| \geq 3R^{1/2}$, and that

$$\Delta^2 \psi_R = \frac{16|x|^4}{R^3} \psi'''' \left(\frac{|x|^2}{R} \right) + \frac{16|x|^2(d+2)}{R^2} \psi''' \left(\frac{|x|^2}{R} \right) + \frac{4d(d+2)}{R} \psi'' \left(\frac{|x|^2}{R} \right).$$

We then get the bound

$$\begin{aligned} (IV) &\leq \|\Delta^2 \psi_R\|_{L^\infty(|x| \leq 2R^{1/2})} \int \gamma^{(1)}(t, x, x) dx \\ &\lesssim R^{-1} \int \gamma^{(1)}(0, x, x) dx \end{aligned} \quad (4.18)$$

where we have used Proposition 2.1 to obtain the last equality. We may then choose R sufficiently large so that

$$(IV) \leq 4|E_1|. \quad (4.19)$$

Combining (4.9) with (4.12), (4.17) and (4.19), we obtain

$$\partial_{tt} \operatorname{Tr}(\psi_R \gamma^{(1)}) \leq 4E_1 + \frac{1}{2}(II) \leq 4E_1$$

for R sufficiently large, where we have used that the identity (4.10) implies $(II) \leq 0$. Since this quantity is independent of t and strictly negative, while $\operatorname{Tr}(\psi_R \gamma^{(1)})$ is strictly positive for all t in the interval of existence, the result follows as before. \square

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